STRESS WAVES IN A COMPOSITE SEMIINFINITE ROD

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We consider the problem of the propagation of stress waves in a rod composed of a viscoelastic portion of finite length and a semiinfinite elastic part when an impulsive load is applied to the end of the rod.

The hereditary properties of the viscoelastic part of the rod are characterized by Yu. N. Rabotnov's kernel and the model of a standard linear body.

Methods of the theory of functions of a complex variable are used to obtain solutions in the form of a sum of stationary and nonstationary parts.

As the kernels in the Boltzmann–Volterra integral relations in the hereditary theory of elasticity the most effective functions from a theoretical and practical standpoint are the fractional-exponential functions due to Yu. N. Rabotnov [1]. The possibility of using such functions in dynamic problems of the theory of linear visco-elasticity was demonstrated in [2].

Of particular interest is the study of stress waves in a rod, the hereditary behavior of which is defined by the $\mathfrak{Э}_\gamma$-function.

Problems concerned with the propagation of stress waves in semiinfinite homogeneous rods have been the subject of study by many authors [3-6].

In this paper we study the stresses in a rod consisting of two parts: a viscoelastic part whose hereditary properties may be described by Yu. N. Rabotnov's relaxation kernel and an elastic part; the stress waves arise through application of a sinusoidal impulsive load to the end of the rod. The solutions are obtained in the form of a sum of stationary and nonstationary parts.

1. We take the x axis along the rod axis, with $x \in [0, l]$ for the viscoelastic part of the rod and $x \in [l, \infty]$ for the elastic part. To the end of the rod we apply the sinusoidal impulsive load $\sigma_0 H(t) \sin\omega t$, where $H(t)$ is the Heaviside unit function.

The resulting stress wave in the viscoelastic medium is reflected and refracted at the boundary $x = l$ separating the parts of the rod. We consider the behavior of the reflected and refracted waves.

The equation of motion has the form

$$E_j u_{,xx} = \rho_j u_{,tt} \qquad (j = 1, 2) \tag{1.1}$$

Here $u = u(x, t)$ is the longitudinal displacement of points of the rod, ρ_j is the density, E_j is the modulus of elasticity of the corresponding medium, the subscript $j = 1$ refers to the viscoelastic part and $j = 2$ to the elastic part; summation by repeated indices is not used here.

The conditions at the junction $(x = l)$ and at the boundary $(x = 0)$ are

$$\begin{gathered}\sigma(x, t) + \sigma_1(x, t) = \sigma_2(x, t), \quad u + u_1 = u_2, \quad x = l \\ \sigma = -\sigma_0 H(t) \sin\omega t, \qquad x = 0\end{gathered} \tag{1.2}$$

The solutions of Eq. (1.1) for the displacements due to the advancing wave U, the wave U_1 reflected in the viscoelastic part, and the wave U_2 refracted into the elastic part may be written in Laplace space in the form

$$U = C\exp(-k_1 x), \qquad U_j = C_j \exp(\pm k_j x) \tag{1.3}$$

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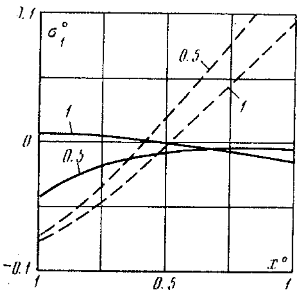

Fig. 1

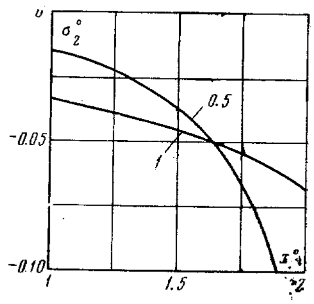

Fig. 2

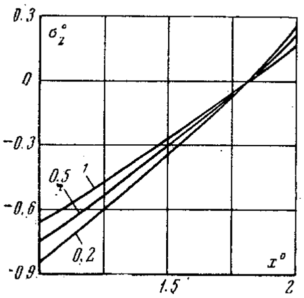

Fig. 3

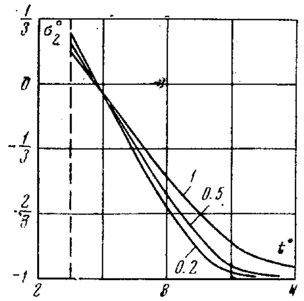

Fig. 4

Here C, C_1, C_2 are coefficients, which play the role of amplitudes, and k_1, k_2 are wave numbers. The plus and minus signs refer to the relected and refracted waves, respectively.

In the Laplace space, for the junction and boundary conditions (1.2), we have

$$\begin{gathered}\sigma(p) + \sigma_1(p) = \sigma_2(p), \quad U + U_1 = U_2, \quad x = l \\ \sigma(p) = -\sigma_0\omega\,(p^2 + \omega^2)^{-1}, \quad x = 0\end{gathered} \tag{1.4}$$

Here p is the complex parameter of the Laplace transform.

Taking note of the conditions (1.4), we may readily determine C, C_1, and C_2 from the expressions (1.3). Recalling the expression for the stress in terms of the displacement

$$\sigma_j(p) = E_j U_{j,x} \quad (j = 1, 2) \tag{1.5}$$

and knowing C, C_1, and C_2, we obtain

$$\begin{gathered}\sigma_1(p) = A\,(E_1k_1 - E_2k_2)\exp\,[k_1(x - 2l)] \\ \sigma_2(p) = -2AE_2k_2\exp\,[-k_1l - k_2(x - l)] \\ A = \sigma_0\omega\,[(p^2 + \omega^2)(E_1k_1 + E_2k_2)]^{-1}\end{gathered} \tag{1.6}$$

Here E_1 and E_2 are moduli of elasticity in complex form.

The wave numbers k_1 and k_2 in complex form may be expressed in terms of the modulus of elasticity of the corresponding media

$$k_1^2 = \rho_1 p^2 E_1^{-1}, \qquad k_2^2 = \rho_2 p^2 E_2^{-1} \tag{1.7}$$

2. Let the hereditary properties of the viscoelastic part of the rod be described by Yu. N. Rabotnov's relaxation kernel, which in Laplace space has the form

$$R(p) = [1 + (p\tau_\varepsilon)^\gamma]^{-1} \quad (0 < \gamma \leqslant 1) \tag{2.1}$$

where τ_ε is the relaxation time. Taking Eq. (2.1) into account, the expressions for E_j and k_j, in accord with Volterra's principle, have the form

$$\begin{gathered}E_1 = E_{\infty 1}[1 - \nu_\varepsilon R(p)], \quad E_2 = E_{\infty 2}, \quad \nu_\varepsilon = (E_{\infty 1} - E_{01})/E_{\infty 1} \\ k_1 = pC_{\infty 1}^{-1}[1 - \nu_\varepsilon R(p)]^{-1/2}, \quad k_2 = C_{\infty 2}^{-1}p\end{gathered} \tag{2.2}$$

Here $E_{\infty 1}$, $E_{\infty 2}$ are the nonrelaxed modulus values and E_{01} is the relaxed modulus value; $C_{\infty j} = (E_{\infty j}/\rho_j)^{1/2}$ are the elastic wave speeds in the corresponding media.

The original functions $\sigma_1(x, t)$ and $\sigma_2(x, t)$ may be determined from the Mellin-Fourier inversion formula

$$\sigma_j(x, t) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \sigma_j(p) e^{pt} dp \qquad (j = 1, 2) \tag{2.3}$$

Substituting the expressions (1.6) into Eq. (2.3) and taking note of the relations (2.1) and (2.2), we obtain

$$\sigma_1(x, t) = \frac{\sigma_0}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \Lambda_1 (\Lambda^{1/2} F - 1) \exp\{\alpha(x - 2l) + pt\} dp$$

$$\sigma_2(x, t) = \frac{\sigma_0}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \Lambda_1 \exp\{-\alpha l - C_{\infty 2}^{-1} p (x - 2l) + pt\} dp \tag{2.4}$$

$$\Lambda = (p^\gamma + a_1)(p^\gamma + a_2)^{-1}, \qquad \Lambda_1 = \omega[(p^2 + \omega^2)(1 + \Lambda^{1/2} F)]^{-1}$$

$$F = C_{\infty 2} E_{\infty 1} / C_{\infty 1} E_{\infty 2}, \qquad \alpha = C_{\infty 1}^{-1} p \Lambda^{-1/2}$$

For $\gamma \neq 1$ the integrand functions in Eqs. (2.4) have a pole $p_k = \pm i\omega$ of the first order, and branch points at $p = 0$ and $p = -\infty$ (the points $p = (-a_1)^{1/\gamma}$, $p = (-a_2)^{1/\gamma}$ are not branch points since for $\gamma \neq 1$ they do not fall on the first sheet of the Riemann surface $|\arg p| < \pi$).

We select a closed contour of integration with a cut along the negative real axis. Using the fundamental theorem on residues and Jordan's lemma, we obtain [7]

$$\sigma_{1\eta}(x, t) = \left[\frac{\sigma_0}{\pi} \int_0^\infty \Omega G_1(x, t, s) ds\right] H(t_1) + \sum_k \operatorname{res} \sigma_1(p_k) e^{p_k t} \tag{2.5}$$

$$\sigma_{2\eta}(x, t) = -\left[\frac{2\sigma_0}{\pi} \int_0^\infty \Omega G_2(x, t, s) ds\right] H(t_2) + \sum_k \operatorname{res} \sigma_2(p_k) e^{p_k t}$$

$$\Omega = \omega(s^2 + \omega^2)^{-1}, \quad t_1 = t + (x - 2l) C_{\infty 1}^{-1}, \quad t_2 = t - l C_{\infty 1}^{-1} - (x - l) C_{\infty 2}^{-1}$$

The functions G_1 and G_2 are defined as follows:

$$G_1 = B_1 \exp[-(\alpha_1 + st)] \{[1 - F^2 R_*] \sin\beta_1 - 2FR_*^{1/2} \sin\varphi \cos\beta_1\} \tag{2.6}$$

$$G_2 = B_1 \exp(\alpha_2 - st) \{[1 + FR_*^{1/2} \cos\varphi] \sin\beta_2 + FR_*^{1/2} \sin\varphi \cos\beta_2\}$$

The quantities appearing in the functions G_1 and G_2 have the form

$$B_1 = [1 + 2FR_*^{1/2} \cos\varphi + F^2 R_*]^{-1}, \qquad \alpha_1 = h_1(x - 2l)\cos\varphi$$

$$R_* = R_1 R_2^{-1} \alpha_2 = h_1 l \cos\varphi + C_{\infty 2}^{-1} s (x - l), \qquad \beta_1 = h_1(x - 2l)\sin\varphi$$

$$\beta_2 = h_1 l \sin\varphi, \qquad h_1 = C_{\infty 1}^{-1} s (R_1/R_2)^{-1/2}, \qquad \varphi = (\varphi_1 - \varphi_2)/2$$

$$R_n = (a_n^2 + 2a_n s^\gamma \cos\delta + s^{2\gamma})^{1/2}, \qquad \operatorname{tg}\varphi_n = (a_n + s^\gamma \cos\delta)^{-1} s^\gamma \sin\delta$$

$$a_1 = (1 - \nu_\varepsilon)\tau_\varepsilon^{-1}, \qquad a_2 = \tau_\varepsilon^{-1}, \qquad \delta = \pi\gamma, \qquad n = 1, 2$$

Here $H(t_1)$ and $H(t_2)$ are Heaviside unit functions.

Putting $\gamma = 1$ into the integrals (2.4), we obtain the stresses in the composite rod, the dynamic behavior of whose viscoelastic part is described by the model of a standard linear body. In this case the integrand expressions in Eqs. (2.4) have the first-order poles $p_k = \pm i\omega$ and the branch points $p = -a_1$, $p = -a_2$. Choosing a closed contour with a cut on the negative real axis from $(-a_1)$ to $(-a_2)$, we obtain

$$\sigma_{1\zeta} = \left[\frac{\sigma_0}{2\pi} \int_{a_1}^{a_2} \Omega G_3(x, t, s) \exp(-st) ds\right] H(t_1) + \sum_k \operatorname{res} \sigma_1(p_k) e^{p_k t} \tag{2.7}$$

$$\sigma_{2\zeta} = \left\{-\frac{2\sigma_0}{\pi} \int_{a_1}^{a_2} \Omega G_4(x, t, s) \exp[C_{\infty 1}^{-1}(x - l)s - st] ds\right\} H(t_2) + \sum_k \operatorname{res} \sigma_2(p_k) e^{p_k t}$$

The functions G_3 and G_4 may be written in the form

$$G_3 = B_2\{(1 - F^2h_2)\sin\psi_1 - 2Fh_2^{1/2}\cos\psi_1\}, \quad h_2 = (s - a_1)(a_2 - s)^{-1}$$
$$G_4 = B_2(h_2^{1/2}F\cos\psi_2 + \sin\psi_2), \quad \psi_1 = C_{\infty1}^{-1}\,\mathrm{sh}_2^{-1/2}(x - 2l) \tag{2.8}$$
$$\psi_2 = C_{\infty1}^{-1}slh_2^{-1/2}, \qquad B_2 = (1 + F^2h_2)^{-1}$$

We remark that the expressions (2.7) cannot be obtained from Eqs. (2.5) by letting $\gamma \to 1$.

Each of the expressions (2.5) and (2.7) is a sum of stationary and nonstationary parts of the stresses for the reflected and refracted waves.

The nonstationary parts are connected with relaxational processes taking place in the dynamical system. In the case of Yu. N. Rabotnov's kernel, the stresses $\sigma_{1\eta}$ for the reflected and $\sigma_{2\eta}$ for the refracted waves, namely, the second terms in the expressions (2.5), can be written in the form

$$\sigma_{j\eta} = A_j\sin(\omega t \pm \theta_j \pm \varkappa_j) \qquad (j = 1, 2) \tag{2.9}$$

The amplitudes of the corresponding waves may be determined by the following relations:

$$A_1 = {}^1/_2\sigma_0q_1[b^2 + (RF^2 - 1)]^{1/2}\exp[\xi(x - 2l)], \quad \xi = C_{\infty1}^{-1}\omega R^{-1/2}\sin\frac{\psi}{2}$$
$$A_2 = \sigma_0q_1\exp(-\xi l), \quad q_1 = \left(1 + RF^2 + 2R^{1/2}F\cos\frac{\psi}{2}\right)^{-1} \tag{2.10}$$
$$b = 2R^{1/2}F\sin\frac{\psi}{2}$$

The quantities appearing in Eqs. (2.10) have the form

$$\mathrm{tg}\,\varkappa_1 = b(RF^2 - 1)^{-1}, \quad \mathrm{tg}\,\varkappa_2 = {}^1/_2b(1 + R^{1/2}\cos\psi/2), \quad \mathrm{tg}\,\psi = d_1d_2^{-1}$$
$$\theta_1 = (x - 2l)d, \quad \theta_2 = ld + C_{\infty2}^{-1}\omega(x - l), \quad d = C_{\infty}^{-1}\omega R^{-1/2}\cos\psi/2$$
$$R = (n_1^2 + n_2^2)^{-1}(d_1^2 + d_2^2)^{1/2}, \quad d_1 = (m_1n_2 - n_1m_1)^2, \quad d_2 = n_1n_2 - m_1^2$$
$$n_k = a_k + \omega^\gamma\cos\delta/2, \quad m_1 = \omega^\gamma\sin\delta/2, \quad \xi = C_{\infty1}^{-1}\omega R^{-1/2}\sin\psi/2 \qquad (k = 1, 2)$$

In the case of the standard linear body the nonstationary parts, namely, the second terms of the expressions (2.7) for the reflected and refracted waves, assume the form

$$\sigma_{j\xi} = D_j\sin(\omega t + \theta_i \pm \varkappa_j) \quad (j = 1, 2;\ i = 3, 4) \tag{2.11}$$

For the amplitude quantities D_j, we have

$$D_1 = {}^1/_2\sigma_0q_2[(rF^2 - 1)^2 + r_1^2]^{1/2}\exp[\xi_1(x - 2l)]$$
$$D_2 = \sigma_0q_2\exp(-\xi_1l), \qquad q_2 = (1 + rF^2 + 2r^{1/2}F\cos\varphi_3/2)^{-1} \tag{2.12}$$
$$\xi_1 = C_{\infty1}^{-1}\omega r^{-1/2}\sin\varphi_3/2, \qquad r_1 = 2r^{1/2}F\sin\varphi_3/2$$

The phase shifts $(\theta_i \pm \chi_j)$ in Eqs. (2.11) may be calculated from the formulas

$$\theta_3 = \lambda_1(x - 2l), \quad \theta_4 = \lambda_1l + C_{\infty2}^{-1}(x - l)\omega, \quad \lambda_1 = C_{\infty1}^{-1}\omega r^{-1/2}\cos\varphi_3/2$$
$$\mathrm{tg}\,\chi_1 = \lambda_2\sin\varphi_3/2, \quad \mathrm{tg}\,\chi_2 = \lambda_3\sin\varphi_3/2, \quad \lambda_2 = 2r^{1/2}F(rF^2 - 1)^{-1}$$
$$\mathrm{tg}\,\varphi_3 = \omega(a_2 - a_1)(a_1a_2 + \omega^2)^{-1}; \quad r = [(a_1^2 + \omega^2)(a_2^2 + \omega^2)^{-1}]^{1/2}$$
$$\lambda_3 = r^{1/2}F(1 + r^{1/2}F\cos\varphi_3/3)^{-1}$$

By way of example, we considered the behavior of the reflected wave with $l = 1$, and the refracted wave on the part $x \in [1; 2]$. Corresponding values of the time t were determined by the Heaviside functions $H(t_1)$ and $H(t_2)$. Calculations were made on an electronic digital computer. In these computations we used the following numerical values for the initial parameters:

$$a_1 = 0.5, \quad a_2 = 1, \quad l = 1, \quad \omega = 1, \quad E_{\infty1} = 0.64, \quad E_{\infty2} = 1, \quad E_{01} = 0.32$$

In Fig. 1 we have drawn, as a function of $x^0 = x(C_{\infty1}\tau_\varepsilon)^{-1}$ $(t^0 = t\tau_\varepsilon^{-1} = 2.5)$, graphs of the quantity $\sigma_1^\circ = \sigma_1\sigma_0^{-1}$ for the stationary (solid curves) and nonstationary (dashed curves) parts of the stress arising in the viscoelastic part of the rod. The labels on the curves indicate values of the fractional parameter γ, which influences the stress distribution as a function of x^0.

Figures 2 and 3 illustrate the behavior of the refracted wave $\sigma_2^\circ = \sigma_2\sigma_0^{-1}$ as a function of x^0 $(t^0 = 2.25)$. The stationary part of the refracted wave (Fig. 2) has the same sign over the length of rod studied at the time that the nonstationary part (Fig. 3) is changing its sign. As the fractional parameter γ decreases, the stresses increase more rapidly in absolute value.

The behavior of the nonstationary part σ_2° as a function of $t^0\,(x^0=2)$ is shown in Fig. 4. The dashed line indicates the instant of wave arrival at the point $x^0=2$. From Fig. 4 it is evident that at the instant of wave arrival the stresses are larger for smaller values of the parameter γ.

Thus the stresses arising in the viscoelastic part of the rod reflect the essential influence of the fractional parameter γ on the stress in the elastic part of the rod.

LITERATURE CITED

1. Yu. N. Rabotnov, "Equilibrium of an elastic medium with after-effect," Prikl. Matem. i Mekhan., 12, No. 1 (1948).
2. S. I. Meshkov, "Integral representation of fractional-exponential functions and their application to dynamical problems of linear viscoelasticity," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 1 (1970).
3. H. Kolsky, Stress Waves in Solids, Dover, New York (1963).
4. D. S. Berry, "A note on stress pulses in viscoelastic rods," Philos. Mag., 3, No. 25 (1958).
5. D. Bland, Theory of Linear Viscoelasticity, Pergamon, New York (1960).
6. V. A. Ivanov, "Theory of propagation of stress waves in a semiinfinite viscoelastic rod," Trudy Leningr. Politekhn. In-ta im. M. I. Kalinina, No. 278 (1967).
7. M. L. Rasulov, Methods of Contour Integration, Am. Elsevier (1967).